5 Cauchy–Riemann equations

5.1 Cauchy–Riemann equations

Recall that we call $f\colon E \longrightarrow \mathbf{C}$ *holomorphic* in domain E, if it is differentiable at every point in E. We need a simple tool to determine differentiability other then the main definition, which is quite tedious to apply in each particular case. We expect, of course, that complex differentiability must be connected somehow with differentiability of two real valued functions u, v, which we find "inside" our complex function:

$$f(z) = f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y).$$

We start with

Proposition 5.1. *Assume that $f\colon E \longrightarrow \mathbf{C}$ is differentiable at $z = z_0 = x_0 + \mathrm{i}y_0$. Then at this point (x_0, y_0) there exist partial derivatives u'_x, u'_y, v'_x, v'_y and, moreover, they satisfy Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)\,, \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)\,. \tag{5.1}$$

Proof. First I need to show that these partial derivatives exist. I have

$$\frac{f(z + h) - f(z)}{h} \to f'(z)$$

for any h and hence for real ones. For real h, in more details,

$$\frac{u(x + h, y) - u(x, y)}{h} + \mathrm{i}\frac{v(x + h, y) - v(x, y)}{h} \to f'(z).$$

Since the left limit exists, it implies that for both real and imaginary parts limits exist (think this out!) but these limits by definition are u'_x, v'_x. Similarly, assuming that h is pure imaginary, I can show that u'_y, v'_y exist.

Now I will differentiate f with respect to x using the chain rule:

$$\frac{\partial f}{\partial x}(z) = f'(z)\frac{\partial z}{\partial x} = f'(z) = u'_x + \mathrm{i}v_x.$$

Similarly,

$$\frac{\partial f}{\partial y}(z) = f'(z)\frac{\partial z}{\partial y} = \mathrm{i}f'(z) = u'_y + \mathrm{i}v_y.$$

After multiplying the last equality by $-\mathrm{i}$ I must conclude that equations (5.1) hold. ■

It would be great to have a converse statement, but in general the converse is not true (see homework problems). It can be proved (I will give a natural proof somewhat later in our course, a direct proof can be found in the textbook) that

Proposition 5.2. *Assume that u'_x, u'_y, v'_x, v'_y exist, satisfy* (5.1) *and continuous at z_0. Then f is differentiable at z_0.*

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For a little headache: the converse statement to this proposition also false in general!

To show that holomorphic functions are "nice," I state one more result, which proof will be given later.

Proposition 5.3. *An $f\colon E \longrightarrow \mathbf{C}$ is holomorphic in E if and only if u_x, u_y, v_x, v_y exist, continuous, and satisfy* (5.1) *in E.*

Example 5.4. We know that z^2 is entire. Not surprisingly,

$$u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy,$$

and (5.1) hold at every point of $\mathbf{C}$.

Example 5.5. Let $f(z) = \bar{z}^2$. Then

$$u(x,y) = x^2 - y^2, \quad v(x,y) = -2xy,$$

and (5.1) hold only at $x = y = 0$. Hence I can conclude that my f is not differentiable (and hence not holomorphic) at any point $(x,y) \neq (0,0)$, and, by Proposition 5.2, it is differentiable (but still not holomorphic) at the origin.

Here is our first theoretical (quite natural) result that uses (5.1).

Proposition 5.6. *Let E be a domain (i.e., open, connected set) and $f\colon E \longrightarrow \mathbf{C}$ be holomorphic in E and $f'(z) = 0$ for all $z \in E$. Then f must be constant.*

Proof. Since f is assumed to be holomorphic, we have that Cauchy–Riemann equations hold at each point of E and moreover

$$u'_x = u'_y = v'_x = v'_y = 0.$$

This implies that both real functions u, v must be constant: $u(x,y) = A, v(x,y) = B$. Therefore,

$$f(z) = A + \mathrm{i}B = C.$$

∎

Sometimes we know just one part (real or imaginary) of a holomorphic function f. We can always find another one as the following example shows.

Example 5.7. Let

$$u(x,y) = 2e^x \cos y$$

be known. Which function v would make $f = u + \mathrm{i}v$ holomorphic in $\mathbf{C}$?

I have

$$\frac{\partial u}{\partial x} = 2e^x \cos y = (5.1) = \frac{\partial v}{\partial y} \implies v(x,y) = \int 2e^x \cos y \,\mathrm{d}y = 2e^x \sin y + \varphi(x).$$

Moreover,

$$\frac{\partial v}{\partial x} = 2e^x \sin y + \varphi'(x) = (5.1) = -\frac{\partial u}{\partial y} = 2e^x \sin y \implies \varphi'(x) = 0 \implies \varphi(x) = C.$$

If we fix one value of our f, e.g., $f(0) = 2$, we can uniquely determine

$$f(z) = f(x + \mathrm{i}y) = 2e^x \cos y + 2\mathrm{i}e^x \sin y.$$

5.2 Geometric meaning of $f'(z)$ and conformal mappings

Recall that $f\colon E \longrightarrow \mathbf{C}$ geometrically a map from one plane to another. Assume that we have a point M in plane z and a half-line emanating from this point. In plane w I will have point $M^*=f(M)$ and a curve to which this half-line being mapped. Let N be a point on the half-line at distance ρ from M and N^* be its image under f. Consider the ratio of the corresponding distances

$$\frac{M^*N^*}{MN}=\frac{1}{\rho}\sqrt{(u(x+\rho\cos\theta,y+\rho\sin\theta)-u(x,y))^2+(v(x+\rho\cos\theta,y+\rho\sin\theta)-v(x,y))^2}\,.$$

Take the limit $\rho\to 0$. Assuming that u,v are differentiable, we'll get

$$\lambda=\lim_{\rho\to 0}\frac{M^*N^*}{MN}=\sqrt{(u'_x\cos\theta+u'_y\sin\theta)^2+(v'_x\cos\theta+v'_y\sin\theta)^2}\,.$$

Quantity λ is called *linear magnification ratio*. When it does not depend on θ?. Rewriting

$$\lambda^2=A\cos^2\theta+2B\cos\theta\sin\theta+C\sin^2\theta,$$

where

$$A=(u'_x)^2+(v'_x)^2,\quad B=u'_xu'_y+v'_xv'_y,\quad C=(u'_y)^2+(v'_y)^2,$$

I obtain (fill in the details) that λ does not depend on θ if and only if

$$A=C,\quad B=0.$$

Now let me calculate the angle between MN and M^*N^*. For MN I have

$$\mu=\tan\theta,$$

for M^*N^* it is

$$\frac{v(x+\rho\cos\theta,y+\rho\sin\theta)-v(x,y)}{u(x+\rho\cos\theta,y+\rho\sin\theta)-u(x,y)}\,,$$

which tends to

$$\tan\varphi=\nu=\frac{v'_x\cos\theta+v'_y\sin\theta}{v'_x\cos\theta+v'_y\sin\theta}=\frac{v'_x+v'_y\mu}{u'_x+u'_y\mu}\,,$$

as $\rho\to 0$.

Now

$$\xi=\tan(\varphi-\theta)=\frac{\nu-\mu}{1+\nu\mu}=\frac{v'_x+(v'_y-u'_x)\mu-u'_y\mu^2}{u'_x+(u'_y+v'_x)\mu+v'_y\mu^2}\,.$$

φ is called a rotation of f at M with respect to a given half-line. This rotation will not depend on θ if and only if

$$v'_x=\xi u'_x,\quad (v'_y-u'_x)=\xi(u'_y+v'_x),\quad -u'_y=\xi v'_y,$$

where ξ does not depend on θ.

Now to the main point.

Definition 5.8. *Map $f\colon E\longrightarrow\mathbf{C}$ is called conformal in E if at each point $z\in E$ its linear magnification factor and rotation do not depend on the direction θ.*

Almost immediately, I have

Proposition 5.9. *Assume that f is holomorphic in E. Then f is conformal at every point z where $f'(z) \neq 0$.*

Moreover, by carefully analyzing the given conditions the student can conclude that the converse also holds! Therefore, being conformal is a characterization of holomorphic functions.

Now you can see the main geometric property of conformal (or holomorphic) maps: they do not change angles between any two curves under this map. Indeed, since rotation does not depend on the direction θ, any half-line at a given point will be rotated by the same angle φ and hence the angle between two curves will not change.

Remark 5.10. We can also ask a natural question: Which maps have only λ not depending on θ? I will leave it as an exercise to show that in this case we'll end up either with holomorphic f, for which (5.1) holds, or with a map for which $u'_x = -v'_y, u'_x = v'_y$. These are not Cauchy–Riemann equation, and they will hold for functions of the form $s = f(\bar{z})$, where f is holomorphic. Thus geometrically we also have a reflection with respect to the real axis, such maps are called *anti-conformal*: they keep the angles, but change the orientation.

Now we can state the geometric meaning of complex derivative. Assume that f is holomorphic in E. Hence I have (it does not depend on θ, hence I can take $\theta = 0$)

$$\begin{aligned} \lambda &= \sqrt{(u'_x)^2 + (v'_x)^2} = |f'(z)|, \\ \varphi &= \arctan \xi = \arg f'(z), \end{aligned}$$

or in words, the modulus of the derivative geometrically is a linear magnification ratio, and the argument of the derivative (assuming $f'(z) \neq 0$) is the rotation at a given point.

Example 5.11. Jumping a little ahead let me take

$$f(z) = \frac{1}{z}\,.$$

Since, as expected,

$$\left(\frac{1}{z}\right)' = -\frac{1}{z^2}\,,$$

then

$$|f'(z)| = \frac{1}{r^2}\,, \quad \arg f'(z) = \pi - 2\arg f(z),$$

which indicates that figures close the origin are magnified by the action of f, far from the origin — reduced.

5.3 More about differentiability

5.4 A glimpse of Wirtinger calculus